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Longest Cycles in Triangle-Free Graphs

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We prove that, for each longest cycle in a 2-connected triangle-free graph G of minimum valence δ on at most $6\delta - 6$ vertices, $G - C$ has at most one edge. It follows that such a graph contains a longest cycle which is also dominating.

1. INTRODUCTION

We consider only finite simple graphs. Given a graph G and a subgraph K of G, $V(K)$ denotes the vertex set of K, and the subgraph of G induced by $V(G) - V(K)$ is denoted by $G - K$ or by $G - V(K)$. We abbreviate $|K| := |V(K)|$. Cycles and paths in G are considered as subgraphs of G. In particular, $|Q| := |V(Q)|$ for a path Q. A cycle C in G is dominating if $G - C$ has no edge. While considering a graph G and its subgraphs, δ denotes the minimum valence of G.

In this paper we shall prove

THEOREM 1.1. *Let C be a longest cycle in the 2-connected triangle-free graph G. If $|G| \leqslant 6\delta - 6$, then $G - C$ cannot have two edges.*

In fact, we will separately prove the following two propositions.

PROPOSITION 1.2. *Let C be a longest cycle in the 2-connected triangle-free graph G. If $G - C$ contains a component on at least three vertices, then $|G| \geqslant 6\delta - 5$.*

PROPOSITION 1.3. *Let C be a longest cycle in the connected triangle-free graph G. If $G - C$ has two distinct components each of which contains exactly two vertices, then $|C| \geqslant 6\delta - 6$ and $|G| \geqslant 6\delta - 2$.*

R. Häggkvist [5] conjectured that a 2-connected bipartite graph on at most 6δ vertices is Hamiltonian. P. Ash and B. Jackson [2] showed that if

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G is a 2-connected bipartite graph and each set in the bipartition has at most $3\delta-3$ vertices, then G contains a longest cycle which is also dominating. Later, P. Ash [1, Theorem 2.1(i)] further proved that given such a graph G, $G-C$ consists entirely of components H such that $|H| \leqslant 2$.

As a consequence of Theorem 1.1, we obtain

COROLLARY 1.4. *A 2-connected triangle-free graph G with $|G| \leqslant 6\delta-6$ contains a longest cycle which is also dominating.*

The examples in [2] show that the upper bound $6\delta-6$ in Theorem 1.1 cannot be improved to $6\delta-4$, and that a longest cycle in a bipartite graph on $n \geqslant 4\delta+4$ vertices is not necessarily dominating.

In a forthcoming paper, Theorem 1.1 will be used in proving that every longest cycle in a 2-connected δ-regular bipartite graph on at most $6\delta-6$ vertices is dominating.

We mention that the existence of cycles C in a graph G such that $G-C$ has only small components was studied by H. J. Veldman [8, 9], P. Fraisse [4], and J. A. Bondy and G. Fan [3], though the cycles constructed in those papers are not necessarily longest cycles, and that the existence of a (not necessarily longest) dominating cycle in a 2-connected triangle-free graph G with $|G| \leqslant 6\delta-5$ can easily be deduced from a theorem of H. J. Veldman [8, Theorem 3].

2. PRELIMINARIES

Let G be a graph, u and v distinct vertices of G, and K a subgraph of G.

We shall write "$u \sim v$" to indicate that u and v are adjacent. If H is a subgraph of G then the set $\{w \in V(K): w \sim y \text{ for some } y \in V(H)\}$ will be denoted by $N_K(H)$ or $N_K V(H)$. Abbreviate $|N_K(v)|$ to $d_K(v)$ and $d_G(v)$ to $d(v)$. A uv-path is a path joining y and v. If $\{u, v\} \subset V(K)$ and K is connected, then a longest uv-path P with $V(P) \subset V(K)$ will be denoted by $L_K(u-v)$.

If P is a path in G and H is a component of $G-P$, then P^H denotes the longest subpath of P with terminal vertices in $N_P(H)$. For vertices a, b on P, let $P[a, b]$ denote the subpath of P with terminal vertices a and b (with the order from a to b if necessary).

Let C be a cycle in G and x, y two distinct vertices on C. After designating the orientation of C, $[x, y]$ denotes the segment (path) of C joining x and y such that the orientation of the segment from x to y is the same as that of C; $[x, y)$ and (x, y) denote the paths $[x, y]-y$ and $[x, y]-\{x, y\}$, respectively.

LEMMA 2.1. *Let $Q = Q[u, w]$ be a uw-path, in a triangle-free graph, where $u \neq w$. Let v be a vertex on Q. Then,*

(a) $|Q| \geqslant 2d_Q(u)$,

(b) $|Q| \geqslant 2d_Q(v) - 1$,

(c) $|Q| \geqslant d_Q(u) + d_Q(v)$,

(d) $|Q| \geqslant 2d_Q(v)$ *if* $v \notin N_Q(u) \cap N_Q(w)$,

(e) $|Q| \geqslant 2d_Q(v) + 1$ *if* $v \notin N_Q\{u, w\}$,

(f) $|Q| \geqslant 2d_Q(u) + 1$ *if* $u \notin N_Q(w)$.

Proof. (a) can easily be checked, and (b)–(f) are consequences of (a). ∎

The proof of the following lemma is partially based on that of Theorem 2.1 in [6].

LEMMA 2.2. *Let a and b be distinct vertices and P a longest ab-path in the 2-connected triangle-free graph G. Each component H of $G - P$ has a vertex v such that $|P^H| \geqslant 2d(v) - 1$.*

Proof. By induction on $|G|$. Let H be a component of $G - P$, and label $N_P(H) = \{x_1, x_2, ..., x_s\}$ according to the order on P from a to b. Since G is 2-connected, $s \geqslant 2$. Since P is a longest ab-path, x_{i+1} is not the successor of x_i. Therefore, $|P^H| \geqslant 2s - 1 + |P[x_i, x_{i+1}]| - 3$ for any $i < s$. Since the result is immediate if $|H| = 1$, we assume that $|H| \geqslant 2$.

If $G - P^H \neq H$, then the subgraph H^* of G induced by $V(H) \cup V(P^H)$ is a 2-connected triangle-free proper subgraph of G in which P^H is a longest $x_1 x_s$-path. By the induction hypothesis, $|P^H| \geqslant 2d_{H^*}(v) - 1 = 2d(v) - 1$ for some $v \in V(H)$.

If H contains a cut vertex, choose a longest path $Q = Q[c_1, c_2]$ in H whose terminal vertices are cut vertices of H (may be $c_1 = c_2$). Further determine two distinct components K_1 of $H - c_1$ and K_2 of $H - c_2$ such that K_1 and K_2 contain no vertex of Q. Since G is 2-connected, $N_P(K_1) \neq \varnothing \neq N_P(K_2)$. Notice that, for $h = 1$ and 2, the subgroup K_h^* of G induced by $V(K_h) \cup \{c_h\}$ is a triangle-free block.

Label $N_P(K_1 \cup K_2) = \{z_1, z_2, ..., z_q\}$ according to the order on P from a to b.

First assume $q \geqslant 2$. There is some $j < q$ such that $z_j \in N_P(K_1)$ and $z_{j+1} \in N_P(K_2)$ or else $z_j \in N_P(K_2)$ and $z_{j+1} \in N_P(K_1)$, say $w_1 \in N_{K_1}(z_j)$ and $w_2 \in N_{K_2}(z_{j+1})$. Let Q_h be a longest $w_h c_h$-path in H, for $h = 1, 2$. If $V(K_h) \subset V(Q_h)$ then, by Lemma 2.1(a), $|Q_h| \geqslant 2d_H(w_h) \geqslant d_H(w_h) + 1$. If $V(K_h) - V(Q_h) \neq \varnothing$ then $|Q_h| \geqslant 2d_{K_h^*}(v) - 1 = 2d_H(v) - 1 \geqslant d_H(v) + 1$ for some $v \in V(K_h^* - Q_h)$ by the induction hypothesis. In any case we can choose a vertex $v_h \in V(K_h)$ such that $|Q_h| \geqslant d_H(v_h)+1$, for $h=1$ and 2. Thus, $|P^H| \geqslant 2q-1+|P[z_j, z_{j+1}]|-3 \geqslant 2q-1+|Q_1 \cup Q \cup Q_2|-1 \geqslant 2q-1+|Q_1|+|Q_2|-2 \geqslant d_P(v_1)+d_P(v_2)-1+d_H(v_1)+d_H(v_2)=d(v_1)+d(v_2)-1$.

Assume, now, $q=1$. Since $s=|N_P(H)| \geqslant 2$, there is some $z \in N_P(H)-\{z_1\}$. Pick $x \in N_{K_1}(z_1)$ and $y \in N_H(z)$. Then $y \notin V(K_1) \cup V(K_2)$. Let Q_1 be a longest xc_1-path in H and Q_2 a c_1y-path in H. As we have shown, one can choose a vertex $v \in V(K_1)$ such that $|Q_1| \geqslant 2d_H(v)-1$; then, $|P^H| \geqslant |P[z_1, z]| \geqslant 2+|Q_1 \cup Q_2| \geqslant 2+|Q_1| \geqslant 2(1+d_H(v))-1 \geqslant 2d(v)-1$.

Suppose, now, that $G-P^H=H$ and H has no cut vertex. Then, $a=x_1$ and $b=x_s$.

Case 1. $s \geqslant 3$. Since G is 2-connected and $|H| \geqslant 2$, there is some $x_i \in N_P(H)$ such that $|N_H\{x_i, x_{i+1}\}| \geqslant 2$. Let G^* be the subgraph of G induced by $V(H) \cup V(P[x_i, x_{i+1}])$. Then, G^* is a 2-connected triangle-free proper subgraph of G in which $P[x_i, x_{i+1}]$ is a longest x_ix_{i+1}-path. By the induction hypothesis, $|P[x_i, x_{i+1}]| \geqslant 2d_{G^*}(v)-1$ for some $v \in V(H)$. Consequently, $|P^H| \geqslant 2s-1+|P[x_i, x_{i+1}]|-3 \geqslant 2(s-2)+2d_{G^*}(v)-1 \geqslant 2dG-G^*(v)+2d_{G^*}(v)-1=2d(v)-1$.

Case 2. $s=2$. Since $|H| \geqslant 2$ and G is 2-connected, one can choose two distinct vertices $u \in N_H(a)$ and $w \in N_H(b)$ in such a way that $|L_H(u-w)|$ is maximum. Let Q be a longest uw-path in H.

First assume $|Q|=|H|$. Since $|Q| \geqslant 2$ and G is triangle-free, there is some $v \in V(Q)$ such that $d_P(v) \leqslant 1$. By Lemma 2.1(b), $|Q| \geqslant 2d_Q(v)-1=2d_H(v)-1$. Hence, $|P| \geqslant 2+|Q| \geqslant 2+2d_H(v)-1 \geqslant 2d(v)-1$.

Assume, now, $V(H)-V(Q) \neq \varnothing$. Clearly, $|Q| \geqslant 3$ and hence $|P^H| \geqslant 5$. Form a new graph G^* by removing all the vertices of $V(P)-\{a, b\}$ from G, adding a new vertex, and joining it to a and b. Then, G^* is a 2-connected triangle-free graph and $|G^*|<|G|$. And $Q^* := aQ[u, w]b$ is a longest ab-path in G. By the induction hypothesis, $|Q^*| \geqslant 2d_{G^*}(v)-1$ for some $v \in V(H-Q)$. Since $|P^H| \geqslant |Q^*|$ and $d_{G^*}(u)=d_G(u)$ for every $u \in V(H)$, $|P^H| \geqslant 2d(v)-1$. ∎

3. Proof of Proposition 1.2

Let G be a 2-connected triangle-free graph. Suppose that for some longest cycle C' in G, there is a component H', in $G-C'$, such that $|H'| \geqslant 3$. Among all pairs C', H' such that C' is a longest cycle and H' is a component of $G-C'$ with $|H'| \geqslant 3$, we choose one for which $|H'|$ is as small as possible, and denote it by C, H.

We shall prove this proposition by a discussion on the structure of H. First we shall discard easily the case when $|H|=3$, then treat in two separate proofs the case where H is 2-connected, and the case where H admits a cut-vertex.

Choose an otientation of C and fix it throughout the proof.

Since $|H| \geqslant 3$ and G is 2-connected, $|G| \geqslant 9$; assume, therefore, $\delta \geqslant 3$.

CLAIM 3.1. *If* $|H| = 3$, *then* $|G| \geqslant 6\delta - 5$.

Proof. Suppose $|H| = 3$. Then, H is a path (star) on three vertices, for it is triangle-free. Let v_1 and v_2 be the two terminal vertices and u the middle vertex of H. Label $N_C\{v_1, {}_2\} = \{y_1, y_2, ..., y_r\}$ in the orientation around C (the subscripts of y will be reduced modulo r). Notice that $r \geqslant 2$ since $\delta \geqslant 3$. One can easily check, using the maximality of $|C|$, that

$$|[y_i, y_{i+1})| \geqslant 2 + 2d_{(y_i, y_{i+1})}(u) \qquad \text{for every } y_i \tag{1}$$

and

$$|[y_i, y_{i+1})| \geqslant 4 + 2d_{(y_i, y_{i+1})}(u) \qquad \text{if} \quad N_C\{y_i, y_{i+1}\} = \{v_1, v_2\}. \tag{2}$$

As G is triangle-free, $N_C(u) \cap N_C\{v_1, v_2\} = \varnothing$ and hence $d_C(u) = \sum_{i=1}^{r} d_{(y_i, y_{i+1})}(u)$. Accordingly, using (1) and (2),

$$\begin{aligned}
|C| &= \sum_{i=1}^{r} |[y_i, y_{i+1})| \\
&\geqslant 4\,|N_C(v_1) \cap N_C(v_2)| + 2\,|N_C(v_1) - N_C(v_2)| \\
&\quad + 2\,|N_C(v_2) - N_C(v_1)| + 2d_C(u) \\
&= 2d_C(v_1) + 2d_C(v_2) + 2d_C(u) \\
&= 2d(v_1) + 2d(v_2) + 2d(u) - 8.
\end{aligned}$$

Therefore, $|G| \geqslant |H| + |C| \geqslant 6\delta - 5$. ∎

In view of Claim 3.1, we assume, from now on, that $|H| \geqslant 4$.

Before we check other cases, we shall develop some estimates for $|G|$ in which valences of two distinct vertices are involved.

In the following part, v_1 and v_2 will be two distinct vertices in H which are not cut vertices of H, and Q a path (containing at least two vertices) in H. In connection with v_1, v_2, and Q, the following notation will be used if $N_C(v_1) \neq \varnothing \neq N_C(v_2)$ and $|N_C\{v_1, v_2\}| \geqslant 2$;

$$\begin{aligned}
R_0 &:= \{i \mid y_i \in N_C(v_1) \cap N_C(v_2)\}, \\
R_1 &:= \{i \mid y_i \in N_C(v_1) - N_C(v_2) \text{ and } y_{i+1} \in N_C(v_2)\}, \\
R_{11} &:= \{i \mid \{y_i, y_{i+1}\} \subset N_C(v_1) - N_C(v_2)\},
\end{aligned}$$

and, R_2 and R_{22} are defined by interchanging v_1 and v_2 in R_1 and R_{11}. Note that these sets are disjoint and $d_C(v_h) = |R_0| + |R_h| + |R_{hh}|$ for $h = 1, 2$,

$$r_{hh} := \sum_{i \in R_{hh}} (|[y_i, y_{i+1})| - 3) \qquad \text{for} \quad h = 1, 2,$$

$$r_h := \sum_{i \in R_h} (|[y_i, y_{i+1})| - (|Q| + 1)) \qquad \text{for} \quad h = 0, 1, 2,$$

and

$$q := |Q| + 1 - d_H(v_1) - d_H(v_2).$$

We shall later consider several cases in which we will choose v_1, v_2, and Q in such a way that q, r_0, r_1, r_2, r_{11}, and r_{22} are non-negative and $|H| - |Q| + 3q + r_0 + r_1 + r_2 + r_{11} + r_{22}$ is sufficiently large.

Claim 3.2. (a) *$|[y_i, y_{i+1})| \geqslant 3$ for every $i \in R_{11} \cup R_{22}$.*

(b) *Consequently, $r_{11}, r_{22} \geqslant 0$.*

Proof. Let $i \in R_{hh}$ and $h = 1$ or 2. Since C is a longest cycle, $|[y_i, y_{i+1})| \geqslant 2$. If $|[y_i, y_{i+1})| = 2$ then $C' := v_h[y_{i+1}, y_i] v_h$ is also a longest cycle. As v_h is not a cut vertex of H, $H - v_h$ is a component of $G - C'$. But $|H| > |H - v_h| = |H| - 1 \geqslant 3$ and hence the pair $C', H - v_h$ is better than C, H; this contradicts the choice of C, H. Therefore, $|[y_i, y_{i+1})| \geqslant 3$. ∎

Claim 3.3. *If $|Q| \leqslant |L_H(v_1 - v_2)|$ then*

(a) *$|[y_i, y_{i+1})| \geqslant |Q| + 1$ for every $i \in R_0 \cup R_1 \cup R_2$, and consequently,*

(b) *$r_0, r_1, r_2 \geqslant 0$.*

Proof. For each $i \in R_0 \cup R_1 \cup R_2$, we have a cycle C_i such that $V(C_i) = V([y_{i+1}, y_i]) \cup V(L_H(v_1 - v_2))$ by the definitions of R_0, R_1, and R_2. From $|C_i| \leqslant |C|$ follows $|[y_i, y_{i+1})| \geqslant |L_H(v_1 - v_2)| + 1$. ∎

Claim 3.4. (a) *If*

(i) *$|C| \geqslant 2\,|Q| + 2$,*

(ii) *$|Q| \geqslant 2d_H(v_h) - 1$ for $h = 1, 2$, and*

(iii) *$N_C(v_h) = \varnothing$, then*

$$|C| + |H| \geqslant 6d(v_h) - 1.$$

(b) *If* $|N_C\{v_1, v_2\}| \leqslant 1$, *then*

$$|C| + |H| \geqslant 3d(v_1) + 3d(v_2) - 7 + 3q + (|H| - |Q|) + (|C| - 2|Q| - 2).$$

(c) *If*

(i) $N_C(v_1) \neq \varnothing \neq N_C(v_2)$, *and*

(ii) $v_1 \sim v_2$, *then*

$$|C| + |H| \geqslant 3d(v_1) + 3d(v_2) - 7 + 3q + (|H| - |Q|) + (r_1 + r_2 + r_{11} + r_{22}).$$

(d) *If*

(i) $N_C(v_1) \neq \varnothing \neq N_C(v_2)$.

(ii) $|N_C\{v_1, v_2\}| \geqslant 2$, *and*

(iii) $|Q| \geqslant 5$,

then

$$|C| + |H| \geqslant 3d(v_1) + 3d(v_2) - 13 + 3q + (|H| - |Q|) + (r_0 + r_1 + r_2 + r_{11} + r_{22}).$$

Proof. (a) Since $|H| \geqslant |Q|$ then $|C| + |H| \geqslant 3|Q| + 2$ by (i). By (iii), $d_H(v_h) = d(v_h)$. Then, the conclusion follows by (ii).

(b) Trivially, $|C| + |H| \equiv 3|Q| + 2 + (|H| - |Q|) + (|C| - 2|Q| - 2)$. By the definition of q, $3|Q| + 2 = 3d_H(v_1) + 3d_H(v_2) - 1 + 3q$. For $h = 1$ and 2, $d_H(v_h) \geqslant d(v_h) - 1$ since $|N_C(v_h)| \leqslant 1$. Hence, the assertion.

(c) By (i) and (ii), $R_0 = \varnothing$ (since G is triangle-free), $|N_C\{v_1, v_2\}| \geqslant 2$, and $|R_1| + |R_2| \geqslant 2$. Since $|R_0| = 0$ then $d_C(v_h) = |R_h| + |R_{hh}|$ for $h = 1, 2$. By the definitions of r_1, r_2, r_{11}, and r_{22}, we have $|C| = \sum_{i=1}^{r} |[y_i, y_{i+1})| = (|Q| + 1)(|R_1| + |R_2|) + 3(|R_{11}| + |R_{22}|) + (r_1 + r_2 + r_{11} + r_{22})$. Consequently,

$$|C| + |H| = 3|Q| + 2 + (|Q| + 1)(|R_1| + |R_2| - 2) + 3(|R_{11}| + |R_{22}|) + (|H| - |Q|) + (r_1 + r_2 + r_{11} + r_{22}).$$

By the definition of q, we have $3|Q| + 2 = 3d_H(v_1) + 3d_H(v_2) - 1 + 3q$. From $|Q| + 1 \geqslant 3$ and $|R_1| + |R_2| \geqslant 2$, it follows that $(|Q| + 1)(|R_1| + |R_2| - 2) \geqslant 3(|R_1| + |R_2| - 2)$. Further, $3(|R_1| + |R_2| - 2) + 3(|R_{11}| + |R_{22}|) = 3(|R_1| + |R_{11}|) + 3(|R_2| + |R_{22}|) - 6 = 3d_C(v_1) + 3d_C(v_2) - 6$. We then obtain the required inequality.

(d) From (i) and (ii), it follows that $|R_0|+|R_1|+R_2|\geqslant 2$. By the definitions of $r_0, r_1, r_2, r_{11}, r_{22}$, we have

$$|C|=(|Q|+1)(|R_0|+|R_1|+|R_2|)+3(|R_{11}|+|R_{22}|)$$
$$+(r_0+r_1+r_2+r_{11}+r_{22})$$

and then

$$|C|+|H|=3\,|Q|+2+(|Q|+1)(|R_0|+|R_1|+|R_2|-2)+3(|R_{11}|+|R_{22}|)$$
$$+(|H|-|Q|)+(r_0+r_1+r_2+r_{11}+r_{22}).$$

By the definition of q, $3\,|Q|+2=3d_H(v_1)+3d_H(v_2)-1+3q$. From $|Q|+1\geqslant 6$ (by (iii)) and $|R_0|+|R_1|+|R_2|\geqslant 2$, it follows that $(|Q|+1)(|R_0|+|R_1|+|R_2|-2)\geqslant 6(|R_0|+|R_1|+|R_2|-2)$. Recall also that $|R_0|+|R_h|+|R_{hh}|=d_C(v_h)$ for $h=1, 2$. We now reach the inequality. ∎

Our next task is to find v_1, v_2, and Q which produce a sufficiently large $(|H|-|Q|)+3q+(r_0+r_1+r_2+r_{11}+r_{22})$. The method of choosing v_1, v_2, and Q is based on the method that H. A. Jung used to prove Lemma 3.2 in [6].

I. *Suppose that H Is 2-Connected*

Label $N_C(H)=\{x_1, x_2, ..., x_s\}$ in the orientation around C (the subscripts will be reduced modulo s).

Let $I:=\{i\colon |N_H\{x_i, x_{i+1}\}|\geqslant 2\}$. Since $|H|>1$ and G is 2-connected, then $|I|\geqslant 2$. One can choose, for any $i\in I$, a longest path $Q_i=Q_i[u_i, w_i]$ in H such that $u_i\in N_H(x_i)$ and $w_i\in N_H(x_{i+1})$. Since C is a longest cycle, $|[x_i\, x_{i+1})|\geqslant |Q_i|+1$ for each $i\in I$. Pick $j\in I$ such that $|Q_j|\leqslant |Q_i|$ for every $i\in I$. Note that $|Q_j|\geqslant 3$ since $|H|>2$ and H is 2-connected.

CLAIM 3.5. (a) *If $x_k\neq x_l$ and $|N_H\{x_k, x_l\}|\geqslant 2$ then $|[x_k, x_l)|\geqslant |Q_j|+1$.*

(b) *In particular, $|C|\geqslant 2\,|Q_j|+2$, and, if $Q=Q_j$, then*

(c) *$|[y_i, y_{i+1})|\geqslant |Q|+1$ for every $i\in R_0\cup R_1\cup R_2$, and consequently,*

(d) *$r_0, r_1, r_2\geqslant 0$.*

Proof. Suppose $x_k\neq x_l$ and $|N_H\{x_k, x_l\}|\geqslant 2$. Then, there is some $x_m\in V([x_k, x_l))$ such that $|N_H\{x_m, x_{m+1}\}|\geqslant 2$. By the definition of Q_j, we have $|Q_m|\geqslant |Q_j|$. Hence, $|[x_k, x_l)|\geqslant |[x_m, x_{m+1})|\geqslant |Q_m|\geqslant |Q_j|$.

Since $|N_H\{x_j, x_{j+1})|\geqslant 2$, we obtain $|[x_j, x_{j+1})|$, $|[x_{j+1}, x_j)|\geqslant |Q_j|+1$. Hence, $|C|\geqslant 2\,|Q_j|+2$.

By the definitions of R_0, R_1, and R_2, (c) is a direct consequence of (a), and, by the definitions of r_0, r_1, and r_2, (d) is a direct consequence of (c). ∎

Case 1. $|Q_j| = 3$. Since Q_j is a longest $u_j w_j$-path in H and H is 2-connected, $N_H(v) = \{u_j, w_j\}$ for every $v \in V(H) - \{u_j, w_j\}$.

If $|H| - |Q_j| = 1$, then H is a cycle on four vertices. Choose any two adjacent vertices of H as v_1 and v_2, and $L_H(v_1 - v_2)$ as Q. Since $\delta > 2 = d_H(v_1) = d_H(v_2)$ then $N_C(v_1) \neq \emptyset \neq N_C(v_2)$. And $q = 5 - d_H(v_1) - d_H(v_2) = 1$. Accordingly, $|G| \geqslant 6\delta - 4$ by Claims 3.4(c), 3.2(b), and 3.3(b).

If $|H| - |Q| \geqslant 2$, choose any two distinct vertices from $V(H) - \{u_j, w_j\}$ as v_1 and v_2, and $L_H(v_1 - v_2)$ as Q. Then $|Q| = 5$. By Claim 3.5(b), $|C| \geqslant 2\,|Q_j| + 2 = 8$ and then $|G| \geqslant 13$. Assume, therefore, $\delta \geqslant 4$. Then $d_C(v_1)$, $d_C(v_2) \geqslant 2$, and $|G| \geqslant 6\delta - 13 + (3q + r_0 + r_1 + r_2 + r_{11} + r_{22})$ by Claim 3.4(d). As $\{u_j, w_j\} = N_H(v_1) \cap N_H(v_2)$ and G is triangle-free, $\{x_j, x_{j+1}\} \cap N_C\{v_1, v_2\} = \emptyset$. Suppose $\{x_j, x_{j+1}\} \subset V((y_i, y_{i+1}))$. Since $|L_H(u_j - v_h)| = |L_H(w_j - v_h)| = 4$ for $h = 1, 2$ and C is a longest cycle, $|[y_i, y_{i+1})| = |[y_i, x_j)| + |[x_j, x_{j+1})| + |[x_{j+1}, y_{i+1})| \geqslant 14 = (|Q| + 1) + 8 > 3 + 8$; hence, by Claims 3.2(a), 3.3(a), and by the definitions of r_0, r_1, r_2, r_{11}, and r_{22}, we obtain $r_0 + r_1 + r_2 + r_{11} + r_{22} \geqslant 8$. And $q = 2$ since $|Q| = 5$ and $d_H(v_1) = d_H(v_2) = 2$. Hence, $|G| \geqslant 6\delta + 1$.

Case 2. $|Q_j| = 4$. Label $Q_j = u_j u w w_j$. By Claim 3.5(b), $|C| \geqslant 2\,|Q_j| + 2 \geqslant 10$ which implies that $|G| \geqslant 14$. We, therefore, assume $\delta \geqslant 4$.

Suppose $u \in N_{Q_j}(K_1)$ and $w \in N_{Q_j}(K_2)$ where K_1 and K_2 are components of $H - Q_j$. Since Q_j is a longest $u_j w_j$-path in H and H is 2-connected, $K_1 \neq K_2$, $N_{Q_j}(K_1) = \{u, w_j\}$, and $N_{Q_j}(K_2) = \{u_j, w\}$. But, we then obtain a $u_j w_j$-path which passes through K_2, w, u, and K_1, contradicting the fact that Q_j is a longest $u_j w_j$-path in H. We therefore have $N_{H - Q_j}(u) = \emptyset$ or $N_{H - Q_j}(w) = \emptyset$. Without loss of generality let $N_{H - Q_j}(u) = \emptyset$. Then, $d_H(u) = 2$ because H is triangle-free. Choose u as v_1.

If $N_{H - Q_j}(w) = \emptyset$, choose w as v_2 and Q_j as Q. In this subcase, $q = 1$ and hence Claims 3.4(c), 3.5(d), and 3.2(b) apply to yield $|G| \geqslant 6\delta - 4$.

Let $w \in N_{Q_j}(K)$ where K is a component of $H - Q_j$. Since H is 2-connected and Q_j is a longest $u_j w_j$-path in H, $N_{Q_j}(K) = \{u_j, w\}$ and $|K| = 1$. Choose the vertex of K as v_2, and the path $v_2 u_j v_1 w w_j$ as Q. As $d_C(v_1)$, $d_C(v_2) \geqslant 2$ (since $\delta \geqslant 4$ and $d_H(v_1) = d_H(v_2) = 2$) and $|Q| = 5$, $|G| \geqslant 6\delta - 13 + (3q + r_0 + r_1 + r_2 + r_{11} + r_{22})$ by Claim 3.4(d). Since $u_j \in N_H(v_1) \cap N_H(v_2)$ and $u_j \sim x_j$, $x_j \notin N_C\{v_1, v_2\}$. Suppose $x_j \in V((y_i, y_{i+1}))$. Since H is 2-connected and triangle-free and since Q_j is a longest $u_j w_j$-path in H, there is a $u_j w_j$-path in $H - \{v_2, v_1, w\}$. Hence, $|L_H(v_1 - v_2)| \geqslant 5 = |Q|$. Moreover, by Claim 3.5(a), $|[y_i, y_{i+1})| \geqslant |[y_i, x_j)| + |[x_j, x_{j+1})| \geqslant 2\,|Q_j| + 2 = 10 = (|Q| + 1) + 4 > 3 + 4$; hence, by Claims 3.2(a) and 3.3(a), $r_0 + r_1 + r_2 + r_{11} + r_{22} \geqslant 4$. And, $q = 2$ since $|Q| = 5$ and $d_H(v_1) = d_H(v_2) = 2$. Therefore, $|G| \geqslant 6\delta - 3$.

Case 3. $|Q_j| \geqslant 5$ and $|H| = |Q_j|$. Choose u_j as v_1, the successor of u_j on $Q_j[u_j, w_j]$ as v_2, and Q_j as Q. Since $|Q_j| \geqslant 2d_H(v_2) - 1$ by Lemma 2.1(b), if $N_C(v_2) = \varnothing$ then $|G| \geqslant 6d(v_2) - 1$ by Claims 3.4(a) and 3.5(b). If $N_C(v_2) \neq \varnothing$ then we obtain $|G| \geqslant 6\delta - 4$ by using Claims 3.4(c), 3.5(d), and 3.2(b) because $q \geqslant 1$ (by Lemma 2.1(c)), $N_C(v_1) \neq \varnothing$, and $v_1 \sim v_2$.

Case 4. $|Q_j| \geqslant 5$ and $H - Q_j$ has distinct components K_1 and K_2. Choose, for $h = 1$ and 2, $v_h \in V(K_h)$ such that $|Q_j^K h| \geqslant 2d_H(v_h) - 1$ by Lemma 2.2. Choose Q_j as Q. Then, $|H| - |Q| \geqslant 2$ and $q = \frac{1}{2}(2\,|Q| + 2 - 2d_H(v_1) - 2d_H(v_2)) \geqslant 0$.

In view of Claims 3.4(a), 3.4(b), and 3.5(b) we assume that $N_C(v_1) \neq \varnothing \neq N_C(v_2)$ and $|N_C\{v_1, v_2\}| \geqslant 2$. Then, by Claim 3.4(d), $|G| \geqslant 6\delta - 11 + (3q + r_0 + r_1 + r_2 + r_{11} + r_{22})$. As $r_0, r_1, r_2, r_{11}, r_{22} \geqslant 0$ (by Claims 3.5(d) and 3.2(b)) and $q \geqslant 0$, it suffices to show that $q \geqslant 2$, or, $q \geqslant 1$ and $r_{11} \geqslant 3$, or $r_0 + r_1 - r_2 + r_{11} + r_{22} \geqslant 6$.

Case 4.1. $x_j \in N_C(v_1)$ and $x_{j+1} \in N_C(v_2)$ (or $x_j \in N_C(v_2)$ and $x_{j+1} \in N_C(v_1)$). By the choice of Q_j, $u_j \notin N_{Q_j}(K_1)$ and $w_j \notin N_{Q_j}(K_2)$. Moreover, if $w_j \in N_{Q_j}(K_1)$, then u_j and the successor of u_j on $Q_j[u_j, w_j]$ cannot be contained in $N_{Q_j}(K_2)$; then, $|Q_j| \geqslant 3 + |Q_j^{K_2}| \geqslant 2d_H(v_2) + 2$ and $|Q_j| \geqslant 1 + |Q_j^{K_1}| \geqslant 2d_H(v_1)$, and hence $q \geqslant 2$. Similarly, if $u_j \in N_{Q_j}(K_2)$ then $q \geqslant 2$. If $\{u_j, w_j\} \cap N_{Q_j}(K_1 \cup K_2) = \varnothing$ then $|Q_j| \geqslant 2 + |Q_j^{K_h}| \geqslant 2d_H(v_h) + 1$ for $h = 1$ and 2; thus, $q \geqslant 2$ again.

Case 4.2. $\{x_j, x_{j+1}\} \subset N_C(v_1) - N_C(v_2)$ (or $\subset N_C(v_2) - N_C(v_1)$). Then, $x_j = y_i$ and $x_{j+1} = y_{i+1}$ for some $i \in R_{11}$. By Claim 3.5(a), $|[x_j, x_{j+1})| \geqslant |Q_j| + 1 \geqslant 6$. Accordingly, $|[y_i, y_{i+1})| - 3 \geqslant 3$. This and Claim 3.2(a) together imply that $r_{11} \geqslant 3$. Moreover, by the choice of Q_j, $\{u_j, w_j\} \cap N_{Q_j}(K_1) = \varnothing$ and hence $|Q_j| \geqslant |Q_j^{K_1}| + 2 \geqslant 2d_H(v_1) + 1$; thus $q \geqslant 1$.

Case 4.3. $x_{j+1} \notin N_C\{v_1, v_2\}$ (or $x_j \notin N_C\{v_1, v_2\}$). Suppose $\{x_j, x_{j+1}\} \subset V([y_i, y_{i+1}))$. By Claim 3.5(a), $|[y_i, y_{i+1})| \geqslant |[x_j, x_{j+1})| + |[x_{j+1}, y_i)| \geqslant 2\,|Q_j| + 2 \geqslant (|Q_j| + 1) + 6 > 3 + 6$; this and Claims 3.2(a), 3.5(c) together imply that $r_0 + r_1 + r_2 + r_{11} + r_{22} \geqslant 6$.

Case 5. $|Q_j| \geqslant 5$ and $H - Q_j$ has only one component K. Choose Q_j as Q. Then, $|H| - |Q| \geqslant 1$.

Case 5.1. $u_j \in N_{Q_j}(K)$ (or $w_j \in N_{Q_j}(K)$). Choose some vertex $v_1 \in V(K)$ such that $|Q_j^K| \geqslant 2d_H(v_1) - 1$ by Lemma 2.2. Choose the successor of u_j on $Q_j[u_j, w_j]$ as v_2. By Lemma 2.1(b), $|Q_j| \geqslant 2d_H(v_2) - 1$. Then, $q \geqslant 0$.

In view of Claims 3.4(a) and 3.5(b), we assume that $N_C(v_h) \neq \varnothing$ for $h = 1$ and 2. Note that $x_j \notin N_C(v_1)$ by the choice of Q_j, and that $x_j \notin N_C(v_2)$ since $v_2 \sim u_j$ and $u_j \sim x_j$.

First assume $|N_C\{v_1, v_2\}| = 1$; that is, $N_C\{v_1, v_2\} = \{y_1\}$. Then, by Claim 3.4(b), $|G| \geqslant 6\delta - 6 + (3q + |C| - 2\,|Q| - 2)$. Suppose $y_1 = x_{j+1}$.

Then, $w_j \notin N_C(K)$ due to the choice of Q_j; hence $|Q_j| \geqslant |Q_j^K| + 1 \geqslant 2d_H(v_1)$ which yields $q \geqslant 1$. Since $|C| - 2|Q| - 2 \geqslant 0$ (by Claim 3.5(b)) and $q \geqslant 1$, $|G| \geqslant 6\delta - 3$. Suppose, $y_1 \neq x_{j+1}$. By Claim 3.5(a), $|C| = |[x_{j+1}, Y_i)| + |[y_i, x_j)| + |[x_j, x_{j+1})| \geqslant 3|Q_j| + 3 \geqslant (2|Q| + 2) + 6$. As $|C| - 2|Q| - 2 \geqslant 6$ and $q \geqslant 0$, $|G| \geqslant 6\delta$.

Assume, now, $N_C(v_1) \neq \varnothing \neq N_C(v_2)$ and $|N_C\{v_1, v_2\}| \geqslant 2$. Suppose $x_j \in V((y_i, y_{i+1}))$ (it was shown that $x_j \notin N_C\{v_1, v_2\}$). By Claim 3.5(b),

$$\begin{aligned} |[y_i, x_{j+1})| &\geqslant |[y_i, x_j)| + |[x_j, x_{j+1})| \geqslant 2|Q_j| + 2 \\ &\geqslant (|Q| + 1) + 6 > 3 + 6. \end{aligned} \tag{3}$$

If $x_{j+1} \in N_C(v_1)$, then $w_j \notin N_{Q_j}(K)$ by the choice of Q_j; hence $|Q_j| \geqslant |Q_j^K| + 1 \geqslant 2d_H(v_1)$ and $q \geqslant 1$. If $x_{j+1} \in N_C(v_2)$ then $v_2 \notin N_H(w_j)$ (since G is triangle-free) and hence $|Q| \geqslant 2d_H(v_2)$ (by Lemma 2.1(d)) from which $q \geqslant 1$ follows. As $q \geqslant 1$, $|H| - |Q| \geqslant 1$, and $r_0 + r_1 + r_2 + r_{11} + r_{22} \geqslant 6$ (by (3) and by Claims 3.2(a), 3.5(c)), $|G| \geqslant 6\delta - 3$ by Claim 3.4(d).

Suppose $x_{j+1} \notin N_C\{v_1, v_2\}$. Then $|[x_{j+1}, y_{i+1})| \geqslant |Q_j| + 1 \geqslant 6$ by Claim 3.5(a). This and (3) together imply that $|[y_i, y_{i+1})| \geqslant (|Q_j| + 1) + 12 > 3 + 12$; that is, $r_0 + r_1 + r_2 + r_{11} + r_{22} \geqslant 12$. By Claim 3.4(d), $|G| \geqslant 6\delta$.

Case 5.2. $\{u_j, w_j\} \cap N_{Q_j}(K) = \varnothing$. Choose u_j as v_1 and w_j as v_2. By Lemma 2.1(a), $|Q| \geqslant 2d_H(v_h)$ for $h = 1$ and 2; hence, $q \geqslant 1$. If $v_1 \sim v_2$ then $|G| \geqslant 6\delta - 3$ by Claims 3.4(c), 3.2(b), and 3.5(d).

Assume $v_1 \notin N_H(v_2)$. Then, by Lemma 2.1(f), $|Q| \geqslant 2d_H(v_h) + 1$ for $h = 1, 2$; hence, $q \geqslant 2$. Since $x_j \sim v_1$ and $x_{j+1} \sim v_2$, $x_j = y_i$ and $x_{j+1} = y_{i+1}$ for some $i \in R_0 \cup R_1$. By Claim 3.5(c), $|[y_i, y_{i+1})| \geqslant |Q| + 1$. If $|[y_i, y_{i+1})| > |Q| + 1$ then $r_0 + r_1 \geqslant 1$ and $|G| \geqslant 6\delta - 5$ by Claims 3.4(d), 3.5(d), and 3.2(b).

Suppose $|[y_i, y_{i+1})| = |Q| + 1$. Then, $[y_{i+1}, y_i]$ and Q form a longest cycle C' in G and the path (y_i, y_{i+1}) is contained in a component H' of $G - C$. According to the choice of C, H, we have $|H'| \geqslant |H| \geqslant |Q| + 1$. But $|(y_{i+1})| = |Q|$. Hence $|H' - (y_i, y_{i+1})| \geqslant 1$. As $V(H' - (y_i, y_{i+1})) \subset V(G - (C \cup H))$, we have $|G - (C \cup H)| \geqslant 1$ and then $|G| \geqslant 6\delta - 5$ by Claims 3.4(d), 3.5(d), and 3.2(b).

II. *Suppose that H Has a Cut Vertex*

Choose two cut vertices c_1 and c_2 of H in such a way that the distance between them is maximum (maybe $c_1 = c_2$). For $h = 1$ and 2, choose a component K_h of $H - c_h$ which does not contain a cut vertex of H. Let the choice be taken so that $K_1 \neq K_2$ and $|K_1| + |K_2|$ is maximum. Denote by K_h^* the subgraph of H induced by $V(K_h) \cup \{c_h\}$, for $h = 1$ and 2. Notice that K_h^* is a triangle-free block. Label $N_C(K_1 \cup K_2) = \{t_1, t_2, ..., t_k\}$ in the orientation around C (the subscripts will be reduced modulo k). Since G is 2-connected, $N_C(K_1) \neq \emptyset \neq N_C(v_2)$ and hence $k \geqslant 1$.

Suppose, first, that $k = 1$. Without loss of generality let $|K_1| \leqslant |K_2|$. Choose a vertex $w_1 \in N_{K_1}(t_1)$ and a longest $w_1 c_1$-path Q_1 in H. Since G is 2-connected there is some $x \in N_C(H) - \{t_1\}$. Suppose $y \in N_H(x)$. Then $y \notin V(K_1 \cup K_2)$. Let Q' be a $c_1 y$-path in H. Clearly $Q_1 \cup Q'$ form a $w_1 y$-path in H. Since C is a longest cycle in G, $|[x, t_1)| \geqslant 1 + |Q_1 \cup Q'| \geqslant 1 + |Q_1|$, and $|[t_1, x)| \geqslant 1 + |Q_1|$ too. Hence, $|C| + |H| \geqslant 2 + 2|Q_1| + |K_1^* \cup K_2^*| \geqslant 4\,|Q_1| + 1$. If $|Q_1| = |K_1^*|$, then $|Q_1| \geqslant 2d_{K_1^*}(w_1) = 2d_H(w_1)$ by Lemma 2.1(a); if $|Q_1| \neq |K_1^*|$ we can choose a vertex $v \in V(K_1^* - Q_1)$ such that $|Q_1| \geqslant 2d_{K_1^*}(v) - 1 = 2d_H(v) - 1$ by Lemma 2.2. Hence, $|G| \geqslant 4\,|Q_1| + 1 \geqslant 8d_H(v) - 3 \geqslant 8d(v) - 11 \geqslant 6d(v) - 5$ for some $v \in V(K_1)$.

Suppose now that $k \geqslant 2$. Let $I := \{i : t_i \in N_C(K_1)$ and $t_{i+1} \in N_C(K_2)$, or $t_i \in N_C(K_2)$ and $t_{i+1} \in N_C(K_1)\}$. Since $k \geqslant 2$, $|I| \geqslant 2$. For each $i \in I$, one can determine a longest $u_i w_i$-path in H such that $u_i \in N_{K_1}(t_i)$ and $w_i \in N_{K_2}(t_{i+1})$, or, $u_i \in N_{K_1}(t_{i+1})$ and $w_i \in N_{K_2}(t_i)$. Pick some $j \in I$ such that $|Q_j| \leqslant |Q_i|$ for every $i \in I$. Assume, without loss of generality, that $u_j \in N_{K_1}(t_j)$ and $w_j \in N_{K_2}(t_{j+1})$. Clearly, Q_j passes through c_1 and c_2, and $Q_j[u_j, c_1]$, $Q_j[c_1, c_2]$, $Q_j[c_2, w_j]$ are longest paths (in H) joining mentioned terminal vertices. Denote these subpaths of Q_j by Q^1, Q^0, and Q^2, respectively.

The following claim is analogous to Claim 3.5(b)–(d) and we therefore omit the proof.

CLAIM 3.6. (a) $|C| \geqslant 2\,|Q_j| + 2$, *and, if* $Q = Q_j$ *and* $v_h \in V(K_h)$ *for* $h = 1$ *and* 2, *then*

(b) $|[y_i, y_{i+1})| \geqslant |Q| + 1$ *for every* $i \in R_0 \cup R_1 \cup R_2$, *and consequently*

(c) $r_0, r_1, r_2 \geqslant 0$. ▮

Case 1. $|Q_j| \geqslant 5$. Choose Q_j as Q. We shall obtain $|G| \geqslant 6\delta - 5$ by Claims 3.4(a), (b), (d), 3.6(a), 3.2(b), and 3.6(c) if we can find vertices $v_1 \in V(K_1)$ and $v_2 \in V(K_2)$ such that (i) $|Q| \geqslant 2d_H(v_h) - 1$ for $h = 1$ or 2, and (ii) $q \geqslant 3$, or, $q \geqslant 2$ and $|H| - |Q| \geqslant 2$.

If $|Q^1| = |K_1^*|$ choose u_j as v_1 and if $|Q^1| \neq |K_1^*|$ choose a vertex $v_1 \in V(K_1 - Q^1)$ such that $|Q^1| \geqslant 2d_H(v_1) - 1$ by Lemma 2.2. If $|Q^2| = |K_2^*|$ choose w_j as v_2 and if $|Q^2| \neq |K_2^*|$ choose a vertex $v_2 \in V(K_2 - Q^2)$ such that $|Q^2| \geqslant 2d_H(v_2) - 1$ by Lemma 2.2. Then, $|Q^h| \geqslant 2d_H(v_h)$ by Lemma 2.1(a) in case $|Q^h| = |K_h^*|$, for $h = 1, 2$.

If $d_H(v_1) = 1$ then $|Q| \geqslant 2d_H(v_1) + 3$ and $|Q| \geqslant |Q^2| + 1 \geqslant 2d_H(v_2)$; hence $q \geqslant 3$. Similarly, $q \geqslant 3$ if $d_H(v_2) = 1$.

Suppose $d_H(v_1)$, $d_H(v_2) \geqslant 2$. Then, $|Q^h| \geqslant 3$ for $h = 1$ and 2. If $v_1 = u_j$, then $|Q| \geqslant |Q^1| + 2 \geqslant 2d_H(v_1) + 2$ by Lemma 2.1(a) while $|Q| \geqslant |Q^2| + 2 \geqslant 2d_H(v_2) + 1$; hence $q \geqslant 3$. Similarly, $q \geqslant 3$ if $v_2 = w_j$. If $v_1 \neq u_j$ and $v_2 \neq w_j$

then $|H|-|Q|\geqslant 2$, and $|Q|\geqslant 2+|Q^h|\geqslant 2d_H(v_h)+1$ (for $h=1,2$) which yields $q\geqslant 2$.

Case 2. $|Q_j|=4$, $|Q^0|=1$. Then, $|Q^1|=3$ and $|Q^2|=2$, or $|Q^1|=2$ and $|Q^2|=3$. Assume, without loss of generality, $|Q^1|=3$ and $|Q^2|=2$. Since K_1^* is a triangle-free block and Q_1 is a longest u_jc_1-path, $V(K_1-Q^1)\neq\varnothing$ and $N_H(v)=\{u_j, c_1\}$ for every $v\in V(K_1)-\{u_j, c_1\}$. Choose a vertex of $V(K_1)-\{u_j, c_1\}$ as v_1 and w_j as v_2 and $L_H(v_1-v_2)$ as Q. Then, $|Q|=5=d_C(v_1)+d_C(v_2)+2$ and hence $q=3$. Since $d_C(v_1)\geqslant 1$, $d_C(v_2)\geqslant 2$ (as $\delta\geqslant 3$), $|Q|=5$, and $q=3$, then Claims 3.4(d), 3.6(c), and 3.2(b) apply to yield $|G|\geqslant 6\delta-4$.

Case 3. $|Q_j|=4$, $|Q^0|\neq 1$. By the choice of c_1, c_2, K_1, K_2, and Q_j, then $c_1\sim c_2$, and, for every $v\in V(H)-\{c_1, c_2\}$, $N_H(v)$ is either $\{c_1\}$ or $\{c_2\}$. Without loss of generality let $d_H(c_1)\leqslant d_H(c_2)$. Then, $|H|\geqslant 2d_H(c_1)$.

Let x, y be two distinct vertices of $N_C\{u_j, c_1, w_j\}$. By the maximality of $|C|$, we find that $|[x, y)|\geqslant 2$ and that $|[x, y)|\geqslant 4$ in case $x\in(N_C(u_j)\cap N_C(w_j))\cup(N_C(w_j)\cap N_C(c_1))$. And, $N_C(u_j)\cap N_C(c_1)=\varnothing$ as $u_j\sim c_1$. Using this observation, we obtain $|C|\geqslant 2d_C(u_j)+2d_C(w_j)+2d_C(c_1)\geqslant 4\delta+2d_C(c_1)-4$; as $|H|\geqslant 2d_H(c_1)$ we further obtain $|G|\geqslant 6\delta-4$.

Case 4. $|Q_j|=3$ (i.e., $c_1=c_2$, $V(K_1)=\{u_j\}$, and $V(K_2)=\{w_j\}$). By the choice of c_1 and c_2, vertex c_1 is the only cut vertex of H. By the maximality condition in the choice of K_1 and K_2, therefore H is a star.

Since C is a longest cycle,

$$|[t_i, t_{i+1})|\geqslant 4+2d_{(t_i, t_{i+1})}(c_1)\geqslant 5+d_{(t_i, t_{i+1})}(c_1)$$

for every i such that $N_{(t_i, t_{i+1})}(c_1)\neq\varnothing$, and, $|[t_i, t_{i+1})|\geqslant 4$ if $\{u_j, w_j\}\subset N_H\{t_i, t_{i+1}\}$. For the remaining segments $[t_i, t_{i+1})$, Claim 3.2(a) (with $\{u_j, w_j\}=\{v_1, v_2\}$) yields $|[t_i, t_{i+1})|\geqslant 3$. Divide the set $\{t_i\in N_C(u_j)\cap N_C(w_j): N_{(t_i, t_{i+1})}(c_1)=\varnothing\}$ into two sets $A:=\{t_i\in N_C(u_j)\cap N_C(w_j): |[t_i, t_{i+1})|=4\}$ and $B:=\{t_i\in N_C(u_j)\cap N_C(w_j): |[t_i, t_{i+1})|\geqslant 5\}$, and put $D:=\{t_i\in N_C(u_j)\cap N_C(w_j): N_{(t_i, t_{i+1})}(c_1)\neq\varnothing\}$. Then,

$$\begin{aligned}|C|\geqslant{}& 4\,|A|+5\,|B|+5\,|D|+3\,|N_C(u_j)-N_C(w_j)|\\&+3\,|N_C(w_j)-N_C(u_j)|+d_C(c_1).\end{aligned}\tag{4}$$

We next show that

$$|G-(C\cup H)|\geqslant|A|.\tag{5}$$

As in Case 5.2 of Part I, one can see that, for each $t_i\in A$, $N_{G-(C\cup H)}((t_i, t_{i+1}))\neq\varnothing$. Assume that x on (t_i, t_{i+1}) and y on (t_l, t_{l+1}) have a common neighbour $v\in V(G-(C\cup H))$, where t_i and t_l are distinct elements of A. Then, the paths $[y, t_i]$, $t_i u_j c_1 w_j t_l$, $[x, t_l]$, and xvy form a cycle C'. From $|C'| \leqslant |C|$ follows $|(t_i, x)| + |(t_l, y)| \geqslant 4$. Similarly, $|(x, t_{i+1})| + |(y, t_{l+1})| \geqslant 2$. We then obtain $|(t_i, t_{i+1})| + |(t_l, t_{l+1})| \geqslant 8$, a contradiction. Hence (5).

By (4) and (5), $|G-H| \geqslant 5(|A|+|B|+|D|) + 3\,|N_C(u_j) - N_C(w_j)| + 3\,|N_C(w_j) - N_C(u_j)| + d_C(c_1)$. From $|A|+|B|+|D| = |N_C(u_j) \cap N_C(w_j)|$ follows $|G-H| \geqslant 3d_C(u_j) + 2d_C(w_j) + d_C(c_1)$. Since $d_H(u_j) = d_H(w_j) = 1$ and $|H| = d_H(c_1) + 1$, $|G| \geqslant 3d(u_j) + 2d(w_j) + d(c_1) - 4 \geqslant 6\delta - 4$.

4. Proof of Proposition 1.3

The observation of H. A. Jung used to prove Lemma 2.7 in [7] leads to this proof.

Let G be a triangle-free graph and C a longest cycle in it. Suppose that $G-C$ contains two distinct components H and H^* with $|H| = |H^*| = 2$. Since the result is trivial for $\delta = 1$, we assume $\delta \geqslant 2$.

Choose an orientation of C and fix it. Label $N_C(H) = \{x_1, x_2, ..., x_s\}$ in the orientation around C. The subscripts will be reduced modulo s. Define

$$X := \{i : V([x_{i-1}, x_i)) \cap N_C(H^*) \neq \varnothing\}$$

and

$$Y := \{i : V([x_{i-1}, x_i)) \cap N_C(H^*) = \varnothing\}.$$

For each $i \in X$, denote by y_i the last vertex of $N_C(H^*)$ on the path $[x_{i-1}, x_i)$. Pick $j \in X$ such that $|(y_j, x_j)| \leqslant |(y_i, x_i)|$ for every $i \in X$. Define, for every i,

$$\sigma_i := |[x_{i-1}, x_i)| - 2\,|N_C(H^*) \cap V([x_{i-1}, x_i))|.$$

As C is a longest cycle, any two consecutive vertices cannot both be in $N_C(H)$ or $N_C(H^*)$. Consequently, $\sigma_i \geqslant 2$ for every $i \in Y$ because $\sigma_i = |[x_{i-1}, x_i)|$ by the definition of Y, and $\sigma_i \geqslant |(y_i, x_i)| - 1$ for every $i \in X$.

We let $V(H) = \{a, b\}$, $V(H^*) = \{u, v\}$, $N_H\{x_j\} = \{a\}$, and $N_{H^*}\{y_j\} = \{u\}$.

Case 1. $|(y_j, x_j)| \geqslant 2$. Then, $\sigma_i \geqslant 1$ for every $i \in X$. Hence, $|C| = \sum_{i=1}^{s} |[x_i, x_{i+1})| \geqslant 2\,|N_C(H^*)| + |X| + 2\,|Y| \geqslant 2\,|N_C(H^*)| + |N_C(H)| \geqslant 6\delta - 6$ and $|G| \geqslant 6\delta - 2$.

Case 2. $|(y_j, x_j)| = 1$. Then $\sigma_i \geqslant 0$ for every $i \in X$. Take any $i \in X - \{j\}$. Let L_i be the longest $x_j x_i$-path and M_i the longest $y_j y_i$-path such that $V(L_i) - \{x_j, x_i\} \subset V(H)$ and $V(M_i) - \{y_j, y_i\} \subset V(H^*)$. Then, $[x_i, y_j]$, M_i,

$[x_j, y_i]$, and L_i form a cycle C'. From $|C'| \leqslant |C|$ follows $|(y_i, x_i)| \geqslant |L_i| + |M_i| - 5$; hence $\sigma_i \geqslant |L_i| + |M_i| - 6$. Accordingly,

$$\sigma_i \geqslant 1 \quad \text{if} \quad y_i \in N_C(u) \quad \text{and} \quad x_i \in N_C(b),$$

or

$$y_i \in N_C(v) \quad \text{and} \quad x_i \in N_C(a),$$

and

$$\sigma_i \geqslant 2 \quad \text{if} \quad y_i \in N_C(v) \quad \text{and} \quad x_i \in N_C(b).$$

Define, for $k \in \{u, v\}$ and $l \in \{a, b\}$,

$$n(kl) := |\{i \in X : y_i \in N_C(k) \text{ and } x_i \in N_C(l)\}|$$

and

$$n(l) := |Y \cap \{i : x_i \in N_C(l)\}|.$$

Then,

$$\begin{aligned} |C| &\geqslant 2\,|N_C(H^*)| + n(ub) + n(va) + 2n(vb) + 2n(a) + 2n(b) \\ &= 2d_C(u) + 2d_C(v) + 2d_C(a) + 2d_C(b) \\ &\quad - (2n(ua) + n(ub) + n(va)). \end{aligned}$$

And $n(ua) + n(ub) \leqslant d_C(u)$ and $n(ua) + n(va) \leqslant d_C(a)$. Therefore, $|C| \geqslant d_C(u) + 2d_C(v) + d_C(a) + 2d_C(b) \geqslant 6\delta - 6$ and $|G| \geqslant 6\delta - 2$.

Case 3. $|(y_j, x_j)| = 0$. Then $\sigma_j \geqslant -1$. Using the argument mentioned in Case 2, one can verify that

$$\sigma_i \geqslant 1 \quad \text{if} \quad y_i \in N_C(u) \quad \text{and} \quad x_i \in N_C(a),$$

$$\sigma_i \geqslant 2 \quad \text{if} \quad y_i \in N_C(u) \quad \text{and} \quad x_i \in N_C(b),$$

or

$$y_i \in N_C(v) \quad \text{and} \quad x_i \in N_C(a),$$

and

$$\sigma_i \geqslant 3 \quad \text{if} \quad y_i \in N_C(v) \quad \text{and} \quad x_i \in N_C(b).$$

Now, one can further compute as in Case 2 to see that $|C| \geqslant 7\delta - 8 \geqslant 6\delta - 6$ and $|G| \geqslant 7\delta - 4 \geqslant 6\delta - 2$.

5. Proof of Corollary 1.4

Let C be a longest cycle in the 2-connected triangle-free graph G with $|G| \leqslant 6\delta - 6$. By Theorem 1.1, it remains to consider the case in which $G - C$ has a unique edge. Let this edge join vertices u and v.

Label $N_C\{u, v\} = \{w_1, w_2, ..., w_n\}$ in some orientation around C (the subscripts will be reduced modulo n). Clearly, $n \geqslant 2\delta - 2$ and $|[w_i, w_{i+1})| \geqslant 2$ for every i. Divide $N_C\{u, v\}$ into two sets $A := \{w_i : |[w_i, w_{i+1})| \geqslant 3\}$ and $B := \{w_i : |[w_i, w_{i+1})| = 2\}$. Then, $|C| \geqslant 3\,|A| + 2\,|B|$, and $N_H\{w_i, w_{i+1}\}$ is either $\{u\}$ or $\{v\}$ for every $w_i \in B$. If $B = \emptyset$ then $|G| \geqslant 3n + 2 \geqslant 6d - 4$, a contradiction. Hence, $B \neq \emptyset$. Denote by v_i be the middle vertex of $[w_i, w_{i+1}]$ for each $w_i \in B$.

As in Case 4 of Part II in the proof of Proposition 1.2, one can see that, for any two distinct vertices w_i and w_j of B, v_i and v_j cannot have common vertices in $G - (C \cup H)$. Therefore, if $N_{G-(C \cup H)}(v_i) \neq \emptyset$ for every $i \in B$, then $|G - (C \cup H)| \geqslant |B|$ and $|G| \geqslant 3\ (|A| + |B|) + 2 = 3n + 2 \geqslant 6\delta - 4$, a contradiction. Hence, $N_{G-(C \cup H)}(v_i) = \emptyset$ for some $i \in B$. Then $[w_{i+1}, w_i]$ and u or v form a longest cycle which is also dominating.

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